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THE INCREASINGLY COMPLEX STRUCTURE OF THE BIFURCATION TREE OF A PIECEWISE-SMOOTH SYSTEM†

M. I. FEIGIN

Nizhnii Novgorod

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A new approach to the study of the dynamics of a piecewise-smooth system is proposed, which uses the a priori known possible bifurcation structures of the parameter space. In Section 1 the synthesis of the structures of the bifurcation tree of the system is considered, namely, the local structures, bifurcation bands, sources and nodes. It is shown that a node corresponding to a doubling bifurcation with reorientation of the domain of existence can generate a sequence of increasingly complex structures. Then the increasing number of unstable orbits serves as one of the mechanisms giving rise to the chaotic behaviour of the dynamical system. In Section 2 the procedure for synthesizing the structures of the bifurcation tree of a piecewise-smooth system proposed in the first part of the paper is applied to the problem of the forced vibrations of a linear oscillator with impacts against a stopping device. Period-doubling cascades are discovered, which are accompanied by the reorientation of the domain of existence of a solution relative to some bifurcation surface, namely, the trunk of the tree. A set of frequency intervals is distinguished on the bifurcation trunk, each containing an infinite sequence of increasingly complex local structures appearing and disappearing at the nodes. This specific mechanism, giving rise to the chaotic motion of the oscillator, is realized in neighbourhoods of the limiting nodal bifurcation points.

1. THE SYNTHESIS OF BIFURCATION TREE STRUCTURES

Elementary local bifurcation structures. We shall consider piecewise-smooth dynamical systems described by equations of the form

$$dx/dt = f(x, t, \mu) \tag{1.1}$$

where x is the n-dimensional coordinate vector, f is a vector-valued function periodic in t, and μ is a parameter vector.

A trajectory in an $(n+1)$-dimensional phase space G corresponds to the variation of the state $x(t)$ of the system with time. The domain G is split into subdomains $G_1, G_2, \ldots, G_j$ in which the different subsystems are defined, each described by its own Eq. (1.1) with a sufficiently smooth right-hand side. The phase space trajectories of the subsystems are spliced together in some way at the boundaries of the domains.

It proves convenient to reduce the analysis of a piecewise-smooth system to the analysis of the Poincaré mapping $\Pi(x)$ of the boundaries of $G_1, \ldots, G_j$ into themselves ($x_{n+1} \equiv t$). A stationary point $x_* = \Pi(x_*)$ of the mapping will then correspond to a periodic solution.

In what follows, when studying the dependence of the solutions of a piecewise-smooth system on the parameters we shall consider two kinds of bifurcations. The first is the same as in analytic systems. It corresponds to the loss of stability and occurs whenever an eigenvalue of the Jacobian matrix $\Pi'(x_*)$ or a root of the characteristic equation

$$\chi(\lambda, \mu) = 0 \tag{1.2}$$

moves out of the unit circle. In this case $\lambda = 1$, or $\lambda = -1$, or $\lambda = \exp(\pm i\varphi)$, $i = \sqrt{(-1)}$, $0 < \varphi < \pi$. The corresponding bifurcation surfaces will be denoted by N_+, N_-, N_φ.

In general, when the parameter being varied crosses the bifurcation boundary, it proves convenient to describe the bifurcation in terms of local structures. Within the family of seven possible elementary N-bifurcation structures [1–4], we restrict ourselves to the confluence of a stable and an unstable solution on N_* (Fig. 1a)

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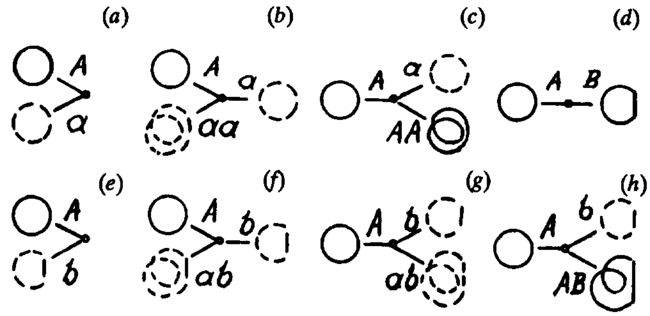

Fig. 1.

$$A, a \rightleftarrows \phi \tag{1.3}$$

and two structures involving the loss of stability on N_- accompanied either by the confluence with an unstable doubly winding solution (Fig. 1b)

$$A, aa \rightleftarrows a \tag{1.4}$$

or the appearance of a stable doubly winding solution (Fig. 1c)

$$A \rightleftarrows a, AA \tag{1.5}$$

The C-bifurcations [3, 5, 6] constitute the other type of periodic solutions of piecewise-smooth systems. They involve a change in the number of pieces from which the closed phase space trajectory (orbit), corresponding to the solution under consideration, is spliced together.

We assume that in the limiting C-bifurcation setting the orbits of two solutions A (or a) and B (or b) are the same, the largest number of spliced pieces corresponding to the orbits B and b.

Let $\chi_a(\lambda, \mu)$, $\chi_b(\lambda, \mu)$ and $\chi(\lambda, \mu)$ be the characteristic polynomials whose roots $\lambda = \alpha_i$, $\lambda = \beta_i$, $\lambda = \lambda_i$ determine the stability or instability of the aforesaid solutions and, respectively, the doubly winding orbits AB and ab. It can be shown that the nature of the simplest local C-bifurcation structures depends on whether the following three indices, equal to the number of the corresponding real roots, are odd or even [5, 6]

σ^+—the roots α_i and β_i greater than $+1$;

σ^-—the roots α_i and β_i less than -1;

σ^{++}—the roots α_i^2 and λ_1 greater than $+1$.

We shall restrict ourselves to five local C-bifurcation structures, namely

$$A \rightleftarrows B \tag{1.6}$$

if $\sigma^+ = \sigma^- = 0$ (Fig. 1d)

$$A, b \rightleftarrows \phi \tag{1.7}$$

if σ^+ is odd and σ^- is even (Fig. 1e)

$$A, ab \rightleftarrows b \tag{1.8}$$

if σ^+ is even and σ^- and σ^{++} are odd (Fig. 1f)

$$A \rightleftarrows b, ab \tag{1.9}$$

if σ^- and σ^{++} are even and σ^- is odd (Fig. 1g)

$$A \rightleftarrows b, AB \tag{1.10}$$

if σ^+ is even, σ^- is odd, and $\sigma^{++} = 0$ (Fig. 1h).

Synthesis of a complex local structure. Along with the original set of elementary local structures, additional hypotheses are necessary to turn the synthesis into a specific problem. For example, we shall assume that the difference between the number of stable and unstable stationary points of the Poincaré mapping remains unchanged for a C-bifurcation. In this case the structure (1.9) does not satisfy the assumption made (Fig. 1g). The contradiction can be removed if the local structure is made more complex by a bifurcation giving rise to a stable four-fold winding orbit

$$A \rightleftarrows b, AAAB, ab \tag{1.11}$$

or the confluence with an unstable four-fold winding orbit

$$A, aaab \rightleftarrows b, ab \tag{1.12}$$

The procedure described above enables us to synthesize sequences of increasingly complex local structures. The algorithm of such a complexification can be considered by comparing the structures (1.3), (1.4), as well as (1.7), (1.8), and (1.12). A common feature of the two groups of structures being compared is that the next doubling bifurcation is accompanied by the reorientation of the domain of existence of an unstable "double" orbit with respect to the bifurcation boundary (the trunk). It follows that the sequence of increasingly complex local C-bifurcation structures can be written as

$$A, b \rightleftarrows \phi; \quad A, ab \rightleftarrows b; \quad A, a^3b \rightleftarrows b, ab; \quad A, a^7b \rightleftarrows b, ab, \ a^3b; \ldots$$

A similar sequence can also be formally written for N-bifurcation local structures

$$A, a \rightleftarrows \phi; \quad A, aa \rightleftarrows a; \quad A, a^4 \rightleftarrows a, aa; \quad A, a^3 \rightleftarrows a, aa, a^4; \ldots$$

Synthesis of a bifurcation band. The problem of synthesizing bifurcation transitions containing more than one local structure can be solved by a simple selection of structures from the original set (1.3)–(1.10) if the solutions at the entry and exit of the band are known.

For example, suppose that a dissipative non-autonomous system has a unique solution A for $\mu = \mu_0$, and a finite value μ corresponds to crossing the C-boundary with one of the local structures (1.7), (1.8) or (1.12). By selecting the structures (1.3)–(1.10), we obtain the structures of bifurcation bands L_1, L_2 and L_3 satisfying all the conditions (Fig. 2).

We will consider one more example of the synthesis of a bifurcation band, namely, a bifurcation joint between the orbits AB and b and the orbit B lying in different domains of the parameter space. Two solutions are obviously possible, depending on the local structure introduced into the band: a confluence with a stable orbit of double period or the birth of an unstable doubly winding orbit (Fig. 3). In the former case the bifurcation band L_4 includes three local structures and in the latter L_5 is synthesized from two structures.

Synthesis of a bifurcation source. A structure involving the birth or death of a bifurcation tree will be called a bifurcation source. We consider a family of parametric trajectories $\mu_{10} < \mu_1 < \mu_{11}$, $\mu_2 =$ const and assume that no bifurcations occur when $\mu_2 < 0$, and for $\mu_2 > 0$ the straight line $\mu_2 =$ const intersects

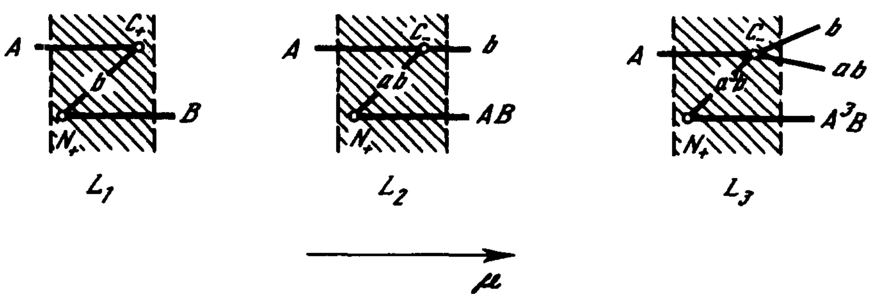

Fig. 2.

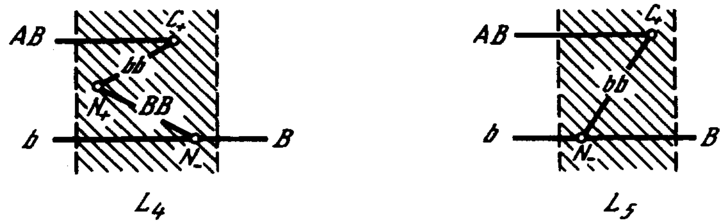

Fig. 3.

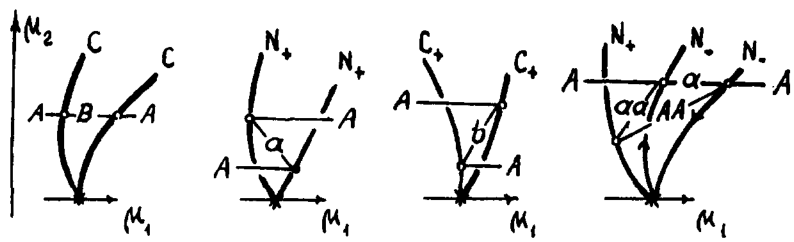

Fig. 4.

the bifurcation boundaries forming the bifurcation band. Obviously, its structure must ensure that the solutions are the same at the entry and exit of the band.

In the examples in Fig. 4 the parametric singular point (source) is indicated by an asterisk. One can talk of a source generating soft bifurcation transitions or safe boundaries, rigid transitions or dangerous boundaries (Fig. 4a), rigid transitions or dangerous boundaries (Fig. 4b, c), and a source giving rise to mixed boundaries (Fig. 4d) [7].

Remarks. 1. The bifurcation sources form folds (simple or multi-sheeted) in a three-dimensional parameter space.

2. For clarity the variations in a system under a rigid transition can be represented in some "hybrid" space. If, in addition to the physical (control) parameters, one also takes as the coordinates some indices characterizing the behaviour of the system (inner parameters), which are functions of the control parameters, then the simplest bifurcation fold will correspond to Whitney's fold [8, 9].

Synthesis of the structure of the trunk of the bifurcation tree. Bifurcation nodes. Next we consider the synthesis of a continuous transition between different local structures when the parameters vary along some boundary or the trunk of a bifurcation tree. We shall show that such structures are separated by a node on the trunk of the tree.

We will confine ourselves to the case of two control parameters such that the C-boundary can be specified in the plane of these parameters. We consider a family of closed parametric trajectories, for example, circles centred on C at a point separating different local structures. Different local structures will then be present at the points of intersection of C and an arbitrary circle.

We take the arc length μ_1 and the radius μ_2 as new control parameters for the family of circles introduced. Setting $\mu_2 = \text{const}$, we choose the initial and final values $\mu_{10} = \mu_{11}$ of the variable parameter in the domain with known solutions. As a result, we arrive at the problem of synthesizing a bifurcation band considered above. Naturally, along with the local structures, the bands L_i already synthesized can now be used as the elements being synthesized (Fig. 5).

We assume that the problem of synthesis has been solved for $\mu_2 > 0$ and that bifurcation transitions between all solutions of the system have been established for $0 < \mu_1 < 2\pi$. Then, using the bifurcation pattern around the circle, which is already known, it remains to explain the deformation it will undergo as $\mu_2 \to 0$. The width of different bifurcation bands on the circle will also tend to zero as $\mu_2 \to 0$. This is so because different bands cannot intersect one another due to the assumption that the local structures are different on either side of the centre of the circle.

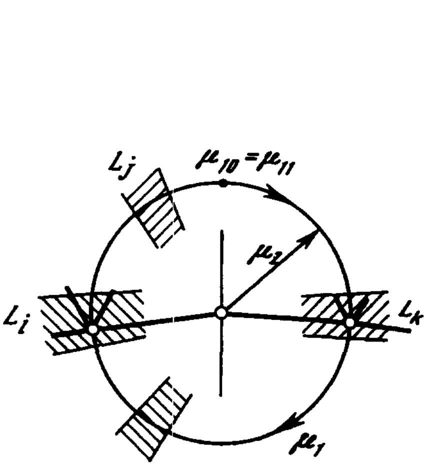

Fig. 5.

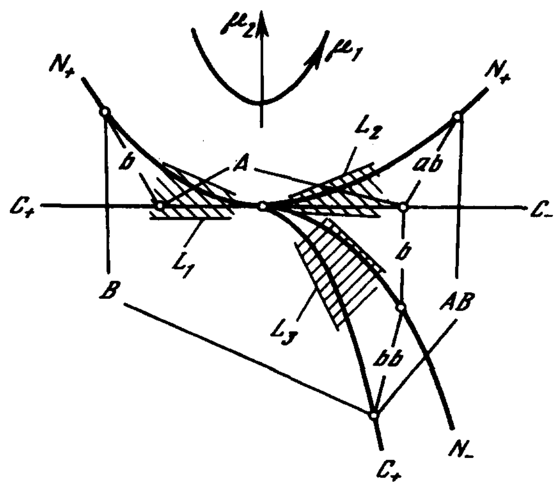

Fig. 6.

The transition from one local structure to another as one moves along the trunk must therefore involve their birth and death at a point, which is a bifurcation node.

In Fig. 6 we show the result of the synthesis of a bifurcation node for the transition between the local structures A, $b \rightleftarrows \phi$ and A, $ab \rightleftarrows b$ as the parameter changes along the C-boundary considered as the trunk of the bifurcation tree. The structure of the node can be constructed using the three bifurcation bands L_1, L_2 and L_3 already synthesized, which are presented in Fig. 2.

Reorientation of the domain of existence of the orbit b with respect to the C-boundary occurs in the case considered above. Therefore, as one moves along the trunk, the local structures must change continuously due to their birth and death at the bifurcation nodes.

Remark. By a special choice of control parameters the bifurcation structures of a node and a source can be deformed into one another. For such a deformation of the node presented in Fig. 6 it suffices to change to control parameters using a family of parabolas, namely, μ_1 along a parabola and μ_2 along the locus of the vertices of the parabolas.

Synthesis of the structure of the trunk of a bifurcation tree, the complexity of which increases without limit. The structure of a node generated by a doubling bifurcation with reorientation of the domain of existence (Fig. 6) has a remarkable feature, namely, it admits of subsequent "alignment" of one component with another. The increasing number of unstable orbits compensating the number of revolutions of the stable orbits, which is doubled every time, represents one of the mechanisms giving rise to the chaotic behaviour of the dynamical system. If the bifurcation structure of a node is represented in its infinitesimal neighbourhood, in which case the width of the bands L_1–L_3 can be neglected, then the trunk of the bifurcation tree will have the structure shown in Fig. 7.

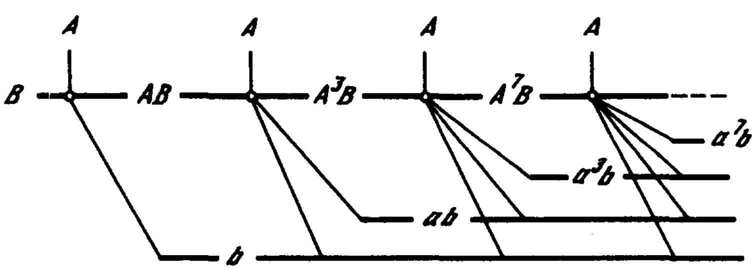

Fig. 7.

2. THE PARAMETRIC PORTRAIT OF AN OSCILLATOR WITH IMPACT AGAINST A STOPPING DEVICE

On the location of the trunk of the bifurcation tree. For the majority of strongly non-linear systems it is impossible to construct all bifurcation boundaries. Along with this, in individual cases a specific regularity is considered in the division of the parameter space, which makes it possible to talk of the characteristic bifurcation tree or the parametric portrait of the dynamical system. There is a class of non-linear characteristics giving rise to a bifurcation picture typical for it. The oscillating system turns out to be robust with respect to this class of characteristics [6, 10].

In this respect, an oscillator with impacts against a stopping device turns out to be quite an effective basic model. The point is that if the stopping device is placed at a distance d greater than the amplitude X of forced oscillations $P(t)$, the system remains linear. This enables us to write down the elementary and, along with this, the exact equation of the trunk of the bifurcation tree in the form

$$d = P(t)_{\max} = X \tag{2.1}$$

No impact-free solution can exist when $x < d$. Equation (2.1) corresponds to the C-bifurcation boundary defining transitions between the linear and non-linear systems. As a result, as soon as the problem is stated, one knows the position in the parameter space of the "scene" in the vicinity of which the basic "bifurcation acts" will take place. Note that even for the Duffing oscillator this problem can only be solved by computer simulation of the original equations.

The domains of existence and stability of subperiodic solutions. We will consider forced oscillations of a linear oscillator with impacts against stopping device described by the equations

$$x'' + 2\nu x' + x = \cos\omega t, \quad x < d \tag{2.2}$$

$$x'^{+} = -Rx'^{-}, \quad x = d \tag{2.3}$$

where d is the distance from the stopping device, ν is the coefficient of viscous friction, x'^{-} and x'^{+} are the velocities before and after the impact, and $R \in (0, 1)$ is the Newtonian coefficient of restitution.

The solution of Eqs (2.2) and (2.3) as the Poincaré mapping $M_1 = \Pi(M_0)$ from the half-plane $x = d, x' < 0$ into itself is spliced together from two parts: M_0M^-, a solution of the linear system (2.2), and M^-M_1, the impact interaction part (2.3) (Fig. 8).

We consider the subperiodic solutions of order n with m impacts within one period $\theta = 2\pi n/\omega$. We denote these by $\Gamma_{m,n}$, distinguishing between stable and unstable solutions $S_{m,n}$ and $U_{m,n}$ if necessary. We set $\nu = 0$ initially, i.e. we consider only dissipativity due to the impacts failing to be entirely elastic, as long as this idealization does not lead to an "unusual" increment of some characteristics of the solution.

The relation between the coordinates of the initial and final points $t_0, x_0 = d, x'_0 < 0$ and $t_1, x_1 = d, x'_1 < 0$ can be written as

$$\begin{aligned} d &= p_1 + (d - P_0)\cos t_{01} + (x'_0 - P'_0)\sin t_{01} \\ -x'_1 / R &= P'_1 - (d - P_0)\sin t_{01} + (x'_0 - P'_0)\cos t_{01} \end{aligned} \tag{2.4}$$

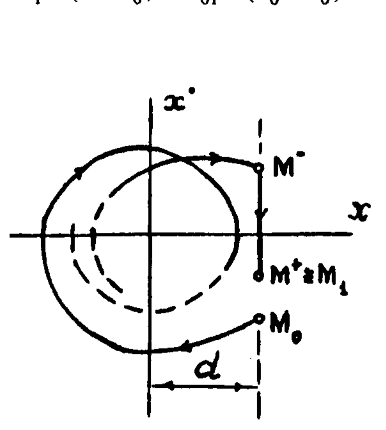

Fig. 8.

$$P_i = \cos \omega t_i / (1 - \omega^2), \quad t_{01} = t_1 - t_0$$

To find the subperiodic solutions $\Gamma_{n,1}$ with one impact within a period we set

$$x_1' = x_0', \quad t_1 - t_0 = 2\pi n / \omega = \theta \tag{2.5}$$

From (2.4) and (2.5) we obtain equations for the coordinates of the stationary point t_0, x_0' of the mapping

$$\begin{aligned} &(1 - z \operatorname{ctg}(\theta / 2))^2 + (r-1)^2 z^2 / (r\omega)^2 = X^2 / d^2 \\ &\sin \omega t_0 = z(1 - r^{-1})(1 - \omega^2) d / \omega \\ &X = |1 - \omega^2|^{-1}, \quad r = (1 + R) / (2R), \quad z = r x_0' / d < 0 \end{aligned} \tag{2.6}$$

To study stability we vary all the phase variables in (2.4) in the neighbourhood of the solution of Eqs (2.4)–(2.6). Setting $\delta x_1' = \lambda \delta x_0'$ and $\delta t_1 = \lambda \delta t_0$, we arrive at the characteristic equation

$$2\lambda^2 - \lambda\{(1+R)^2(\omega^2 s / z + (1 - \omega^2)(1 + c)) - R(1 + c)^2\} + 2R^2 = 0 \tag{2.7}$$

$$s = \sin\theta, \quad c = \cos\theta$$

In the case when (2.7) has complex roots the desired solution is always stable, since $\lambda_1\lambda_2 = R^2 < 1$. Therefore, as the parameters of the system are varied, loss of stability is possible only when $\lambda = 1$ or $\lambda = -1$, i.e. on the bifurcation boundaries N_+ and N_-.

To study the conditions for a solution of a piecewise smooth system to exist one must analyse the C-bifurcation boundaries. In the case under consideration the conditions for splicing the solution together can be reduced to verifying that no additional impacts occur within one period. In the neighbourhood of the C-boundary (2.1) there will be no additional impacts for the solutions $\Gamma_{n,1}$ close to $P(t)$ provided that $n-1$ of the n maximum values of $x(t)$ within one period are less than d. In other words, the following inequalities must be satisfied

$$x(t_j) < d, \quad x'(t_j) = 0, \quad x''(t_j) < 0, \quad j = 1, 2, \ldots, n-1 \tag{2.8}$$

The procedure described above leads to the system of inequalities [11]

$$\begin{aligned} &\mu \sin(2\pi n / \omega) < 0, \quad \mu \sin((n-j)\pi / \omega) \sin(j\pi / \omega) \cos(\pi n / \omega) < 0, \\ &\mu = d / X - 1, \quad |\mu| \ll 1, \quad j = 1, 2, \ldots, n-1 \end{aligned} \tag{2.9}$$

Doubling bifurcation nodes with reorientation of the domain of existence. We denote by $G_{n,m}$ the domain in the parameter space in which the conditions of existence for $\Gamma_{n,m}$ are satisfied. We consider the set of frequency intervals

$$k < 1/\omega < k + 1/n, \quad k = 0, 1, \ldots \tag{2.10}$$

The following assertion holds for the solutions of type $\Gamma_{n,1}$ merging with the solution $P(t)$ of the linear system for $\mu = 0$: in each of the intervals (2.10) there is a sequence of nodal points

$$1 / \omega_i = k + 1 / n_{i+1}, \quad n_{i+1} = 2^i n, \quad i = 1, 2, \ldots \tag{2.11}$$

on the C-boundary $\mu = 0$ that correspond to a doubling bifurcation with the domains $G_{2n,1}, G_{4n,1}, \ldots$ appearing in the half-plane $\mu > 0$ as $1/\omega$ decreases. The reorientation of the domains of existence $G_{2n,1}$, $G_{2n,1}, \ldots$ from the half-plane $\mu > 0$ to the half-plane $\mu < 0$ occurs at the same time.

To prove this assertion we introduce a local parameter ε_i in the neighbourhood of each nodal point (2.11)

$$\varepsilon_i = 1/\omega - 1/\omega_i, \quad -1/n_{i+1} < \varepsilon_i < 1/n_{i+1}, \quad i = 1, 2, \ldots \tag{2.12}$$

Then the period θ_i of $\Gamma_{n_i,1}$ is

$$\theta_i = 2\pi n_i / \omega, \quad i = 1, 2, \ldots \tag{2.13}$$

and the trigonometric functions in (2.9) take the form

$$\sin(2\pi n_i / \omega) = -\sin(2\pi n_i \varepsilon_i), \quad \cos(\pi n_i / \omega) = (-1)^{1+n_i k} \sin(\pi n_i \varepsilon_i)$$
$$\sin(\pi(n_i - j)/\omega) = (-1)^{(n_i - j)k} \sin((n_i - j)(\varepsilon_i + 1/n_{i+1})\pi)$$
$$\sin(j\pi/\omega) = (-1)^{jk} \sin(j\pi(\varepsilon_i + 1/n_{i+1}))$$

Inside the intervals (2.12) containing ε_i

$$2n_i|\varepsilon_i| \leqslant 1, \quad (n_i - j)(\varepsilon_i + 1/n_{i+1}) < 1, \quad j(\varepsilon_i + 1/n_{i+1}) < 1$$

As a result, conditions (2.9) can be reduced to the inequality

$$\mu\varepsilon_i > 0 \tag{2.14}$$

It follows that for $\varepsilon_i = 0$ the domain of existence $G_{n_i,1}$ is reoriented with respect to the μ axis (Fig. 9a).

The complete picture of the domains $G_{n,1}$, $G_{2n,1}$, ... in the interval under consideration can be obtained by superimposing the local pictures (Fig. 9b). When $\mu < 0$, the number of possible solutions with impact increases without limit as the left end of each of the intervals (2.10) is approached. By replacing n by $4n$, $8n$, ..., in (2.11), it can be shown that the reorientation of $G_{4n,1}$, $G_{8n,1}$, ..., respectively, occurs at each of the nodal points (2.11).

The structure of a bifurcation node. We will now investigate $\Gamma_{n_i,1}$ in the vicinity of the nodal points (2.11). For $|\mu|, |\varepsilon_i| \ll 1$ Eqs (2.6) and (2.7) take the form

$$z^2((\pi n_i \varepsilon_i)^2 + (1 - 1/r)^2/\omega^2) + 2z\pi n_i \varepsilon_i + 2\mu = 0 \tag{2.15}$$

$$\lambda^2 - \lambda(2R - \pi n_i \varepsilon_i (1+R)^2 \omega^2 / z) + R^2 = 0 \tag{2.16}$$

In the neighbourhood of a node, from (2.15) we have two stationary points on the C-bifurcation boundary $\mu = 0$ when $\varepsilon_i \neq 0$

$$z_1 = 0, \quad z_2 = -2\pi n_i \varepsilon_i \omega^2 r^2 / (r-1)^2 < 0$$

It follows that there are two solutions for $\varepsilon_i > 0$. The first solution $U_{n_i,1}$, which appears during a C-bifurcation, is unstable, since its characteristic equation implies that $\lambda \to \infty$ as $z_1 \to 0$ for $\varepsilon_i > 0$. The other solution $S_{n_i,1}$, which does not undergo a C-bifurcation at $\mu = 0$, is stable and vanishes as μ increases, merging with the unstable solution $U_{n_i,1}$ on the bifurcation boundary N_{i+}. The equation of N_{i+} can be obtained from (2.15) and (2.16) with $\lambda = 1$. It has the form

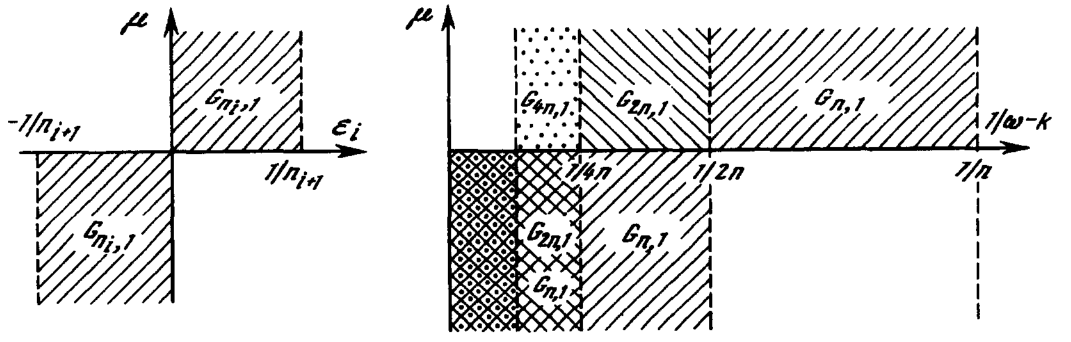

Fig. 9.

$$\mu \approx \frac{(\pi n_i \varepsilon_i)^2 r^2}{2(k+1/n_{i+1})^2 (r-1)^2} \tag{2.17}$$

Note that (2.17) can also be obtained without the characteristic equation (2.16) by equating to zero the discriminant of the quadratic equation (2.15).

In the domain $\varepsilon_i > 0$, $\mu < 0$ the solution $S_{n_i,1}$ is unique. A loss of stability of this solution occurs for $\lambda = -1$ and is accompanied by the birth of the solution $\Gamma_{2n,2}$ with double period.

The substitution of $\lambda = -1$ into (2.16) gives the bifurcation value $z_{-1} = \pi n_i \varepsilon_i \omega^2 < 0$ in the domain $\varepsilon_i < 0$, $\mu < 0$ containing the desired boundary N_{i-}. From (2.15) with $z = z_-$ we have

$$\mu \approx -\frac{(\pi n_i \varepsilon_i)^2}{(k+1/n_{i+1})^2}\left[1+\frac{(r-1)^2}{2r^2}\right] \tag{2.18}$$

The C and N-boundaries found in the neighbourhood of the bifurcation node are represented by the solid lines in Fig. 10(a). In the general case, for oscillating systems with impacts the initial solution is known to lose stability when an additional impact occurs [12]. It can be seen in Fig. 10(a) that the solution $\Gamma_{n_i,1}$ with impacts appearing on the C-boundary $\mu = 0$ is unstable everywhere besides the nodal frequency value denoted by an asterisk. Along with this, it has been shown [13] that the solution with impact that occurs will be stable under the necessary conditions. In the case of an oscillator with impacts the aforesaid condition corresponds to the nodal frequency. This is easy to verify directly from the characteristic equation (2.16). Setting $\varepsilon_i = 0$, we obtain the equation $\lambda^2 - 2\lambda R + R^2 = 0$, the two roots of which are such that $\lambda_1 = \lambda_2 = R < 1$.

We shall now consider the solution $\Gamma_{2n_i,1}$ of double period for $\varepsilon_i < 0$ and note that viscous friction must be taken into account in this case. Repeating the procedure for solving the original equations (2.2) and (2.3) for $\nu \ll 1$ we obtain the following equation for the coordinate x_0' of the stationary point [11]

$$\left(\frac{x_0'}{\omega d}\right)^2\left[r\frac{\exp(-\nu\theta_{i+1})-\cos\theta_{i+1}-\nu\sin\theta_{i+1}}{\operatorname{ch}(\nu\theta_{i+1})-\cos\theta_{i+1}}-1\right]^2+$$
$$+\left[1+\mu-\frac{x_0' r\sin\theta_{i+1}}{d(\operatorname{ch}(\nu\theta_{i+1})-\cos\theta_{i+1})}\right]^2=1 \tag{2.19}$$

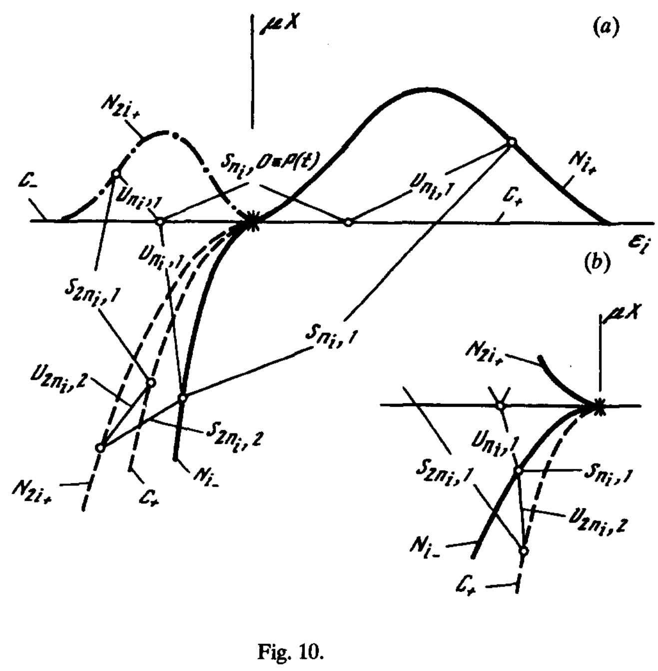

Fig. 10.

On the C-boundary $\mu = 0$, in addition to the root $x_0' = 0$, Eq. (2.19) also has a negative root in the interval $-1/4n_i < \varepsilon_i < 0$, since $\operatorname{sgn}(x_0') = \operatorname{sgn}(\sin \theta_{i+1})$. These roots correspond to two solutions: the unstable solution $U_{2n_i,1}$ appearing during a C-bifurcation and the stable solution $S_{2n_i,1}$. The two solutions merge and disappear on the boundary N_{2i+}. The equation of the latter, which can be found from the condition for the discriminant of the quadratic equation (2.19) to be equal to zero, has the form

$$2\mu \approx \left[\frac{\omega \sin \theta_{i+1}}{(1-\cos\theta_{i+1})(1-1/r)-\nu(\sin\theta_{i+1}+\theta_{i+1})}\right]^2 \tag{2.20}$$

The boundary (2.20) lies above the axis $\mu = 0$, starting and ending at the nodes $\varepsilon_i = -1/4n_i$, $\mu = 0$. In Fig. 10(a) it is represented by the dash–dot line.

To complete the analysis of the structure of the ith node for the n_i and $2n_i$ subperiodic solutions one must make a "bifurcation joint" between the solutions $S_{2nj,1}$ and $U_{ni,1}$ in the domain $\mu < 0$, which exist for $\varepsilon_i < 0$, and the solution $S_{ni,1}$, which exists for $\varepsilon_i > 0$ and has the bifurcation boundary N_- given by (2.18). The synthesis of the corresponding bifurcation band was considered in Section 1 (the structures L_4 and L_5 in Fig. 3). The complete picture of the two versions of the structure of the bifurcation node is shown in Fig. 10. The boundaries introduced for the joint are represented by the dashed lines.

It is interesting to determine the universal constant [4, 14] of the bifurcation sequence as the parameter ω_i or $(1/\omega_i)$ passes through the bifurcation nodes for some initial value of n. In accordance with (2.11), we have

$$\lim_{i\to\infty}(\omega_{i+1}-\omega_i)/(\omega_{i+2}-\omega_{i+1}) = 2$$

Doubling cascades with reorientation. The doubling cascades with reorientation described develop in the intervals (2.10), i.e. $1/\omega \in (k, k+1/n)$ for $k = 0, 1, 2, \ldots$, as the parameters d and ω vary along the C-boundary $\mu = 0$ (or $d = X$). The value of n corresponds to the order of the initial subperiodic solution, which must be chosen to be odd, so that the subintervals already included in the intervals $n/2$ are not considered again.

For each k, to the set of odd n there corresponds the set of intervals (2.10), inside of which the doubling cascades described will develop. These intervals overlap more and more as n increases. The overlapping intervals become condensed as $1/\omega$ decreases, approaching the values $1/\omega = k$, i.e.

$$1/\omega_* = 0, 1, 2, \ldots \tag{2.21}$$

In this case the structure of the trunk $\mu = 0$ of the bifurcation tree becomes extremely complicated, which enables us to talk of the chaotic motion of the oscillator in half-neighbourhoods of the limiting nodal bifurcation points (2.21). The parametric portrait of the oscillator in the plane of $1/\omega$ and μ is shown in Fig. 11. The starting points of cascades corresponding to the initial value $n = 3$ are denoted by an asterisk, the bifurcation boundaries being indicated by a dashed line.

Comparison with the results of a computer simulation. The visualization of the increasingly complex picture of possible motions obtained by computer simulation methods applied to the original system ought to become more practicable as the frequency interval under consideration approaches a resonance frequency. The point is that the dimensions of the domain of parameters along the d-axis will be determined by the values of μX in terms of absolute units. Therefore, bearing in mind the degree of simplicity of visualization of chaotic motion, the limiting nodes should be arranged in the following order: $1/\omega_* = 1, 2, 3, \ldots$.

The theoretical parametric portrait obtained above is consistent with the well-known results of the simulation of forced oscillations of an oscillator with impacts against a stopping device. For example, the half-neighbourhoods of the nodes $1/\omega_* = 1, 2, 3, 4$ are distinguished quite markedly by the complexity of the structure of the parameter plane: a simple two-state regime occurs on one side and limiting complexity on the other [15, Fig. 216a].

The parameter values for which numerous pictures of closed phase space trajectories of solutions of type $S_{n,1}$ (F18, F21–F24, F34, F37–F40, F45, F46, F49–F53) have been obtained [15] lie in the corresponding domains shown in Fig. 11.

In conclusion, we observe that the above results have been obtained by considering the initial single impact motions $\Gamma_{n,1}$. If $\Gamma_{n,2}$, $\Gamma_{n,3}$, ... are considered as the initial solutions, the trunk structure may well become more complex. For example, it turns out that in certain frequency intervals the domains

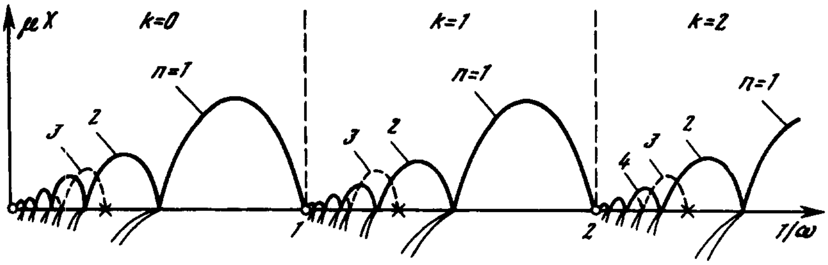

Fig. 11.

of existence of $\Gamma_{n,2}$ alter their orientation with respect to the C-boundary $\mu = 0$, in complete analogy with the case of $\Gamma_{n,1}$ considered above [11].

This paper is dedicated to the memory of Professor N. N. Bautin (1908–1993), an outstanding person and scientist.

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